EPFL

Week 02: Mathematical Modeling Part 1

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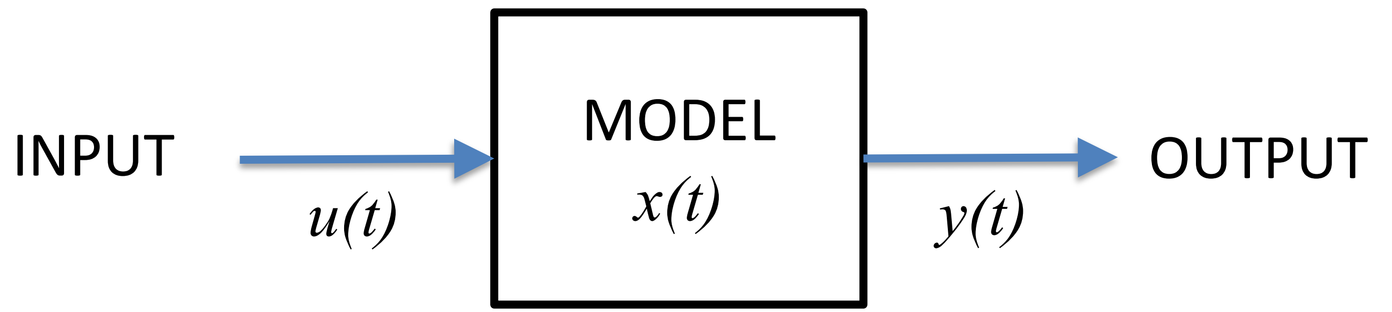
Update

- MATLAB Assignments
  - 1st Assignment is out and due in two weeks
  - You can discuss with your classmates but write the script/report individually
  - Pass or fail for each assignment

- Problem sets
  - Exercise sessions
  - Moodle Forum for each question

Recap

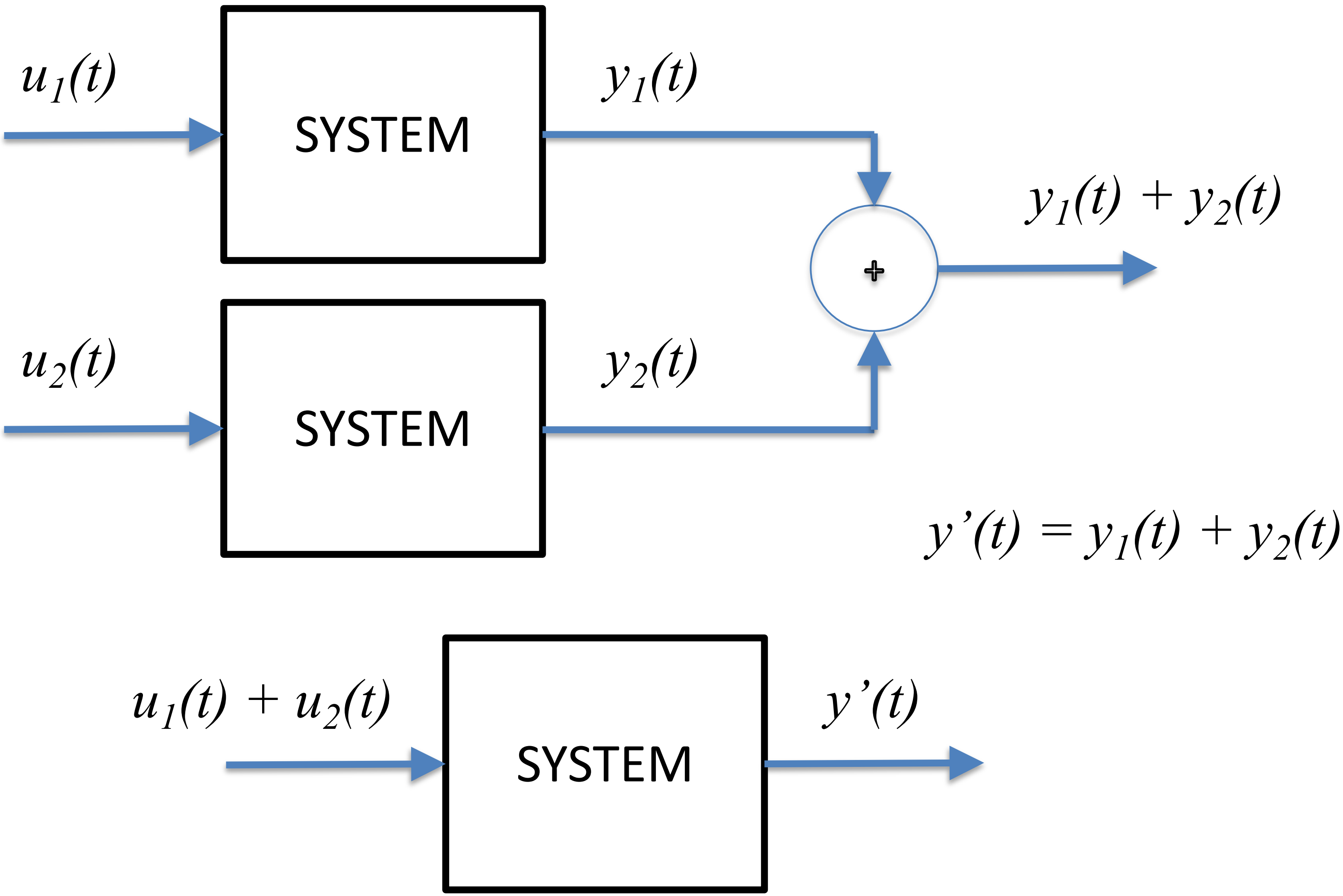
- Definition of System



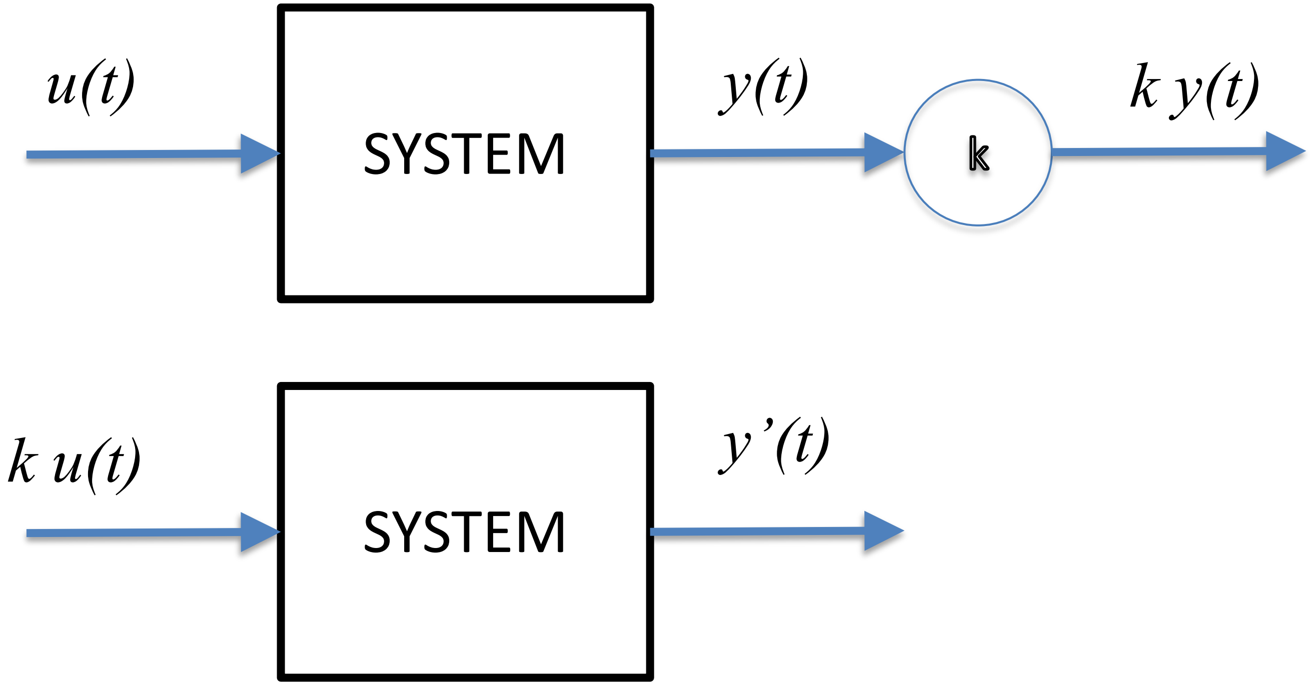
- Parameters and Variables: Input, output, and states.

- Classification of Systems
  - Linear and nonlinear
  - Time-invariant and time-variant

Linear and Nonlinear Systems: Additivity



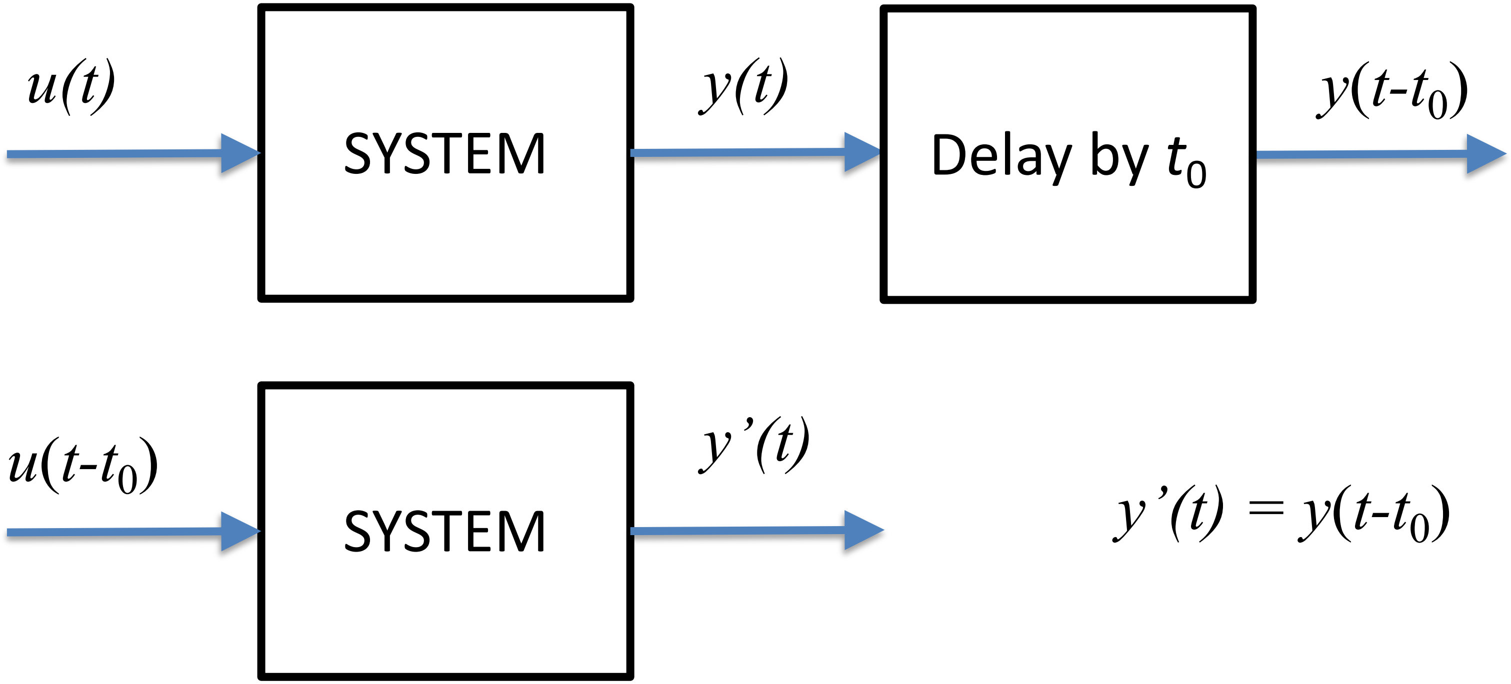
Linear and Nonlinear Systems: Homogeneity



$$y'(t) = k\, y(t)$$

Time Invariance

- From a different perspective the system does not age and the input does not change the foundation of the system
- Mathematically speaking, if input $u(t)$ generates $y(t)$ then the output due to input $u(t\text{-}t_0)$ is $y(t\text{-}t_0)$.



$$y'(t) = y(t\text{-}t_0)$$

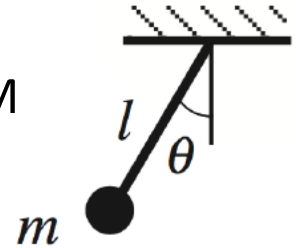
Lecture Overview

- Introduction to modeling procedure
- Mechanical systems
- Electrical systems
- Analogous systems

Building a Representative Mathematical Model

- Structure the problem (main components of the system)
- Use physical laws to derive mathematical equation (Assumptions?)
  - Algebraic equations: static systems
  - Differential equations: dynamical systems
- Use empirical data to determine parameters (static measurements)
- Use empirical data to validate models (dynamic measurements)

SYSTEM

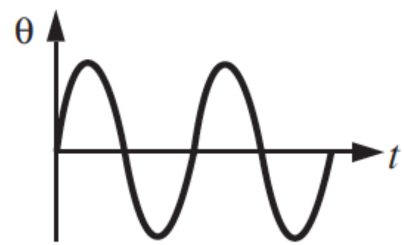


NEWTON'S LAW

$$ml^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mglsin\theta$$

$$\theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \omega_0$$

OBSERVATION



Classical Mechanics

- **General Goal:** Determine what happens to a given set of objects in a given physical situation

- We need to know what makes the objects move the way they do

- Three different strategies
  - Newton's Laws
  - Lagrange's Equations
  - Hamilton's Equations

- They all produce the same information (at the end)

- Conservation Laws (energy, momentum,…)

Motion

- Change in position over time with respect to a frame of reference
- If the position of a body is not changing with respect to a given frame of reference
    - At rest, motionless, stationary, time-invariant position

	Translation	Rotation
Displacement	x	θ
Velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{d^2x}{dt^2}$	$\alpha = \frac{d^2\theta}{dt^2}$

Newton's Laws of Motion

- **First Law**: A body moves with constant velocity (which may be zero) unless acted on by a force
  - INTERTIAL FRAME
- **Second Law:** The time rate of change of the momentum of a body equals the force acting on the body
  - F = ma
- **Third Law:** For every force on one body, there is an equal and opposite force on another body
  - Conservation of momentum
- **Extra:** Forces add up like vectors obeying the principle of superposition.

Newton's Laws of Motion

- **First Law**: Momentum (mv) and angular momentum ($J\omega$) remains constant in the absence of external forces acting on the system
  - m denotes inertial mass and J denotes moment of inertia (angular mass or rotational inertia)

- **Second Law:** The rate of change of momentum of an object is equal to the net applied force.

$$\frac{d}{dt}(mv) = \sum_i F_i$$

$$\frac{d}{dt}(J\omega) = \sum_i M_i$$

SI Units

$F: \left[\text{kg}\,\frac{\text{m}}{\text{s}^2}\right] \equiv [\text{N}]$

$M: [\text{N m}]$

$m: [\text{kg}]$

$J: \left[\frac{\text{kg m}^2}{(\text{rad})}\right] = \left[\frac{\text{N m s}^2}{(\text{rad})}\right]$

$x: [\text{m}]$

$\theta: [(\text{rad})]$

$v: \left[\frac{\text{m}}{\text{s}}\right]$

$\omega: \left[\frac{(\text{rad})}{\text{s}}\right]$

$a: \left[\frac{\text{m}}{\text{s}^2}\right]$

$\alpha: \left[\frac{(\text{rad})}{\text{s}^2}\right]$

Elements of Mechanical Systems

- Force or displacement as an input and motion as an output

- Basic building blocks (idealized)
  - Inertial elements: mass
  - Stiffness elements: springs
  - Damping elements: dampers

- Inertial and stiffness elements store energy
- Damping elements dissipate energy

Inertial Elements

- Masses represent the inertia and resistance to acceleration

$$F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

$$M = J\frac{d^2\theta}{dt^2}$$

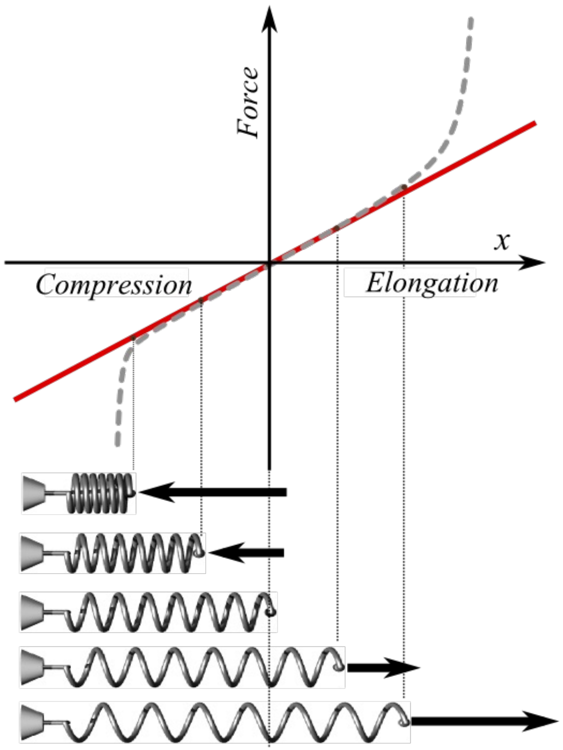
- Moment of Inertia of a rigid body around its axis of rotation

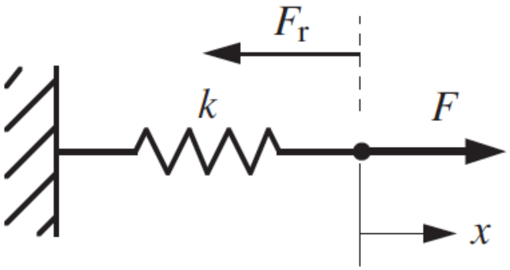
$$J = \int_M r^2 \mathrm{d}m$$

- Simple pendulum: mr^2

Stiffness Elements

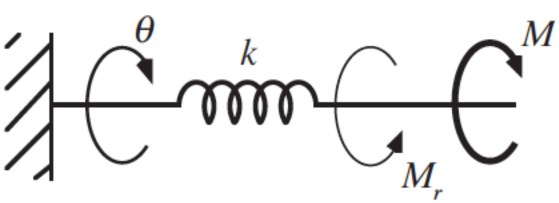
- Energy storage
- **Hooke's Law**: x is small relative to the total possible deformation
- **Linear spring without inertia or damping**: spring coefficient (k) does not change with extension or compression





$$F_r = -F = -kx$$

$$k: \left[\frac{\mathrm{N}}{\mathrm{m}}\right]$$

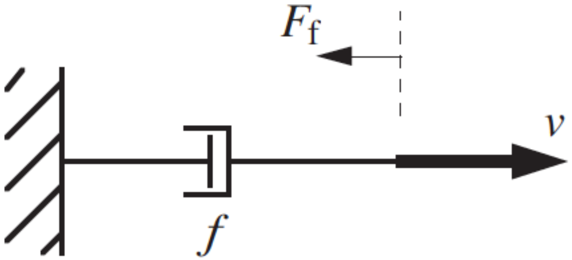


$$M_r = -M = -k\theta$$

$$k: \left[\frac{\mathrm{Nm}}{(\mathrm{rad})}\right]$$

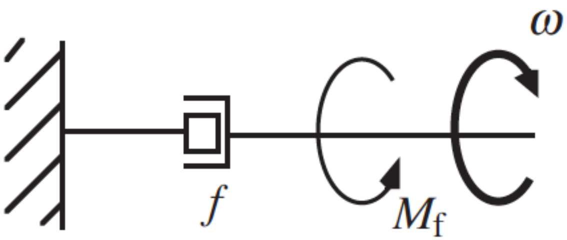
Damping Elements

- Energy dissipation: convert kinetic energy to thermal energy
- Dampers: viscous friction
- Laminar Flow vs Turbulent Flow
- **Linear without inertia and stiffness**: damping coefficient (f) does not change with the magnitude of velocity



$$F_f = -fv = -f\frac{dx}{dt}$$

$$f: \left[\frac{Ns}{m}\right]$$

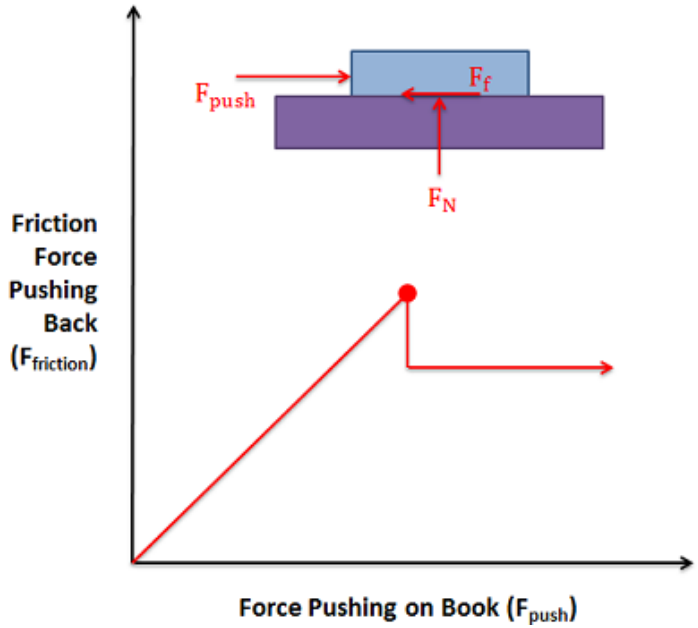


$$M_f = -f\omega = -f\frac{d\theta}{dt}$$

$$f: \left[\frac{Nms}{(rad)}\right]$$

Coulomb Damping

- Coulomb damping absorbs energy via sliding friction
- Friction generated by the relative motion of two surfaces that press against each other
- Static friction: objects are stationary or undergoing no relative motion
- Kinetic friction: objects are sliding against each other



$$F_f = F_{push}$$

Until the point where...

$$F_f = \mu_s * F_N$$

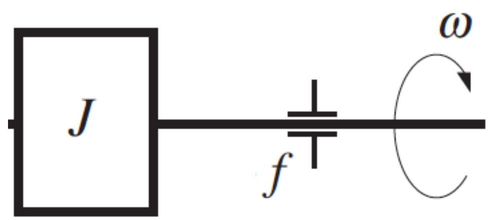
After that point it drops down to and stays at...

$$F_f = \mu_k * F_N$$

Modeling Steps for Mechanical Systems

- Establish an inertial (fixed) coordinate system
- Identify mechanical system elements and isolate them
- Determine minimum number of variables to uniquely define the configuration system
- Draw the free body diagram for each inertial element
- Apply Newton's 2nd Law to each element

Example 1: Motor rotation



Equations of Motion

$$J\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \sum_{\mathrm{i}} M_{\mathrm{i}}$$

$$J\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = -f\omega \qquad \omega(0) = \omega_0$$

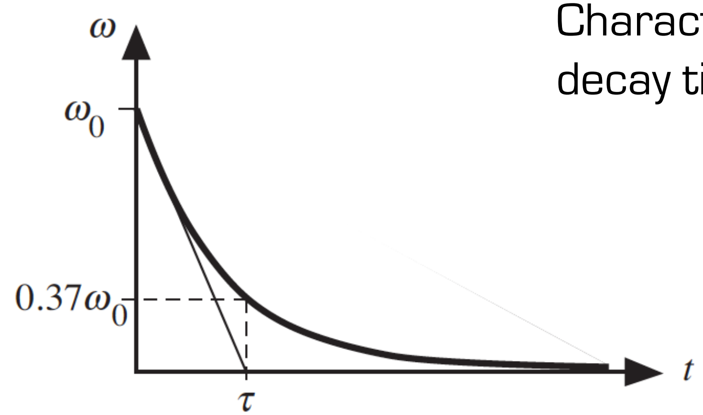
Characteristic Polynomial

$$J\lambda + f = 0 \;\Rightarrow\; \lambda = -\frac{f}{J}$$

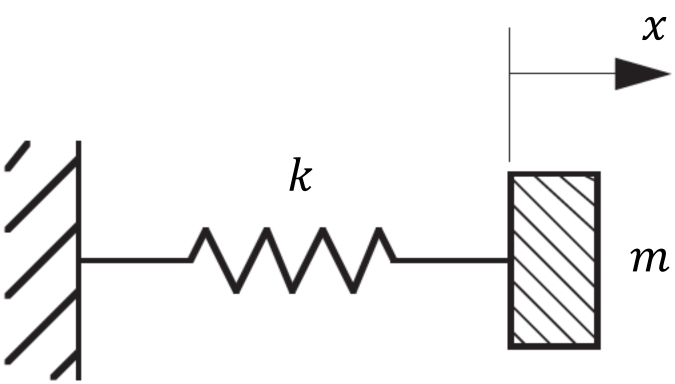
Solution

$$\omega(t) = \omega_0\, \mathrm{e}^{-(\mathrm{f}/\mathrm{J})\mathrm{t}} = \omega_0\, \mathrm{e}^{-\mathrm{t}/\tau} \qquad \tau = \frac{J}{f}$$

Characteristic decay time



Example 2: Mass-spring system



$x(0) = x_0$

$\frac{dx(0)}{dt} = 0$

$x(t) = ?$

Equations of Motion

$$m\frac{d^2x}{dt^2} = -kx \qquad x(0) = x_0 \qquad \frac{dx(0)}{dt} = 0$$

Characteristic Polynomial

$$m\lambda^2 + k = 0 \Rightarrow \lambda_{1,2} = \pm j\sqrt{\frac{k}{m}}$$

Solution

$$x(t) = C_1 e^{j\sqrt{(k/m)}t} + C_1^* e^{-j\sqrt{(k/m)}t}$$

$$\Downarrow \qquad e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

$$x(t) = x_0\cos\left(\sqrt{\frac{k}{m}}t\right)$$

Second Order Linear ODEs

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

If $G(x) = 0$ then $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$

Characteristic Polynomial

$y = e^{rx}$ $\quad ar^2 + br + c = 0$

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$\boldsymbol{b^2 - 4ac > 0}$

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

$\boldsymbol{b^2 - 4ac = 0}$

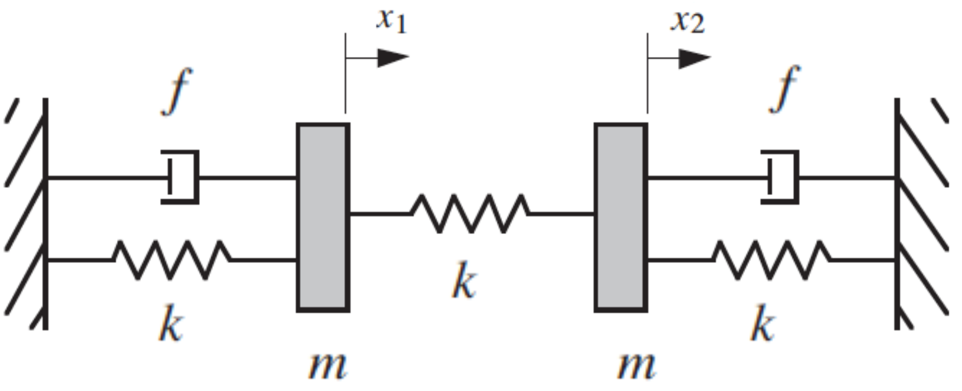
$$y = c_1e^{rx} + c_2xe^{rx}$$

$\boldsymbol{b^2 - 4ac < 0}$

$$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

$$r_1 = \alpha + i\beta$$

Example 3: Mass-spring-damper system



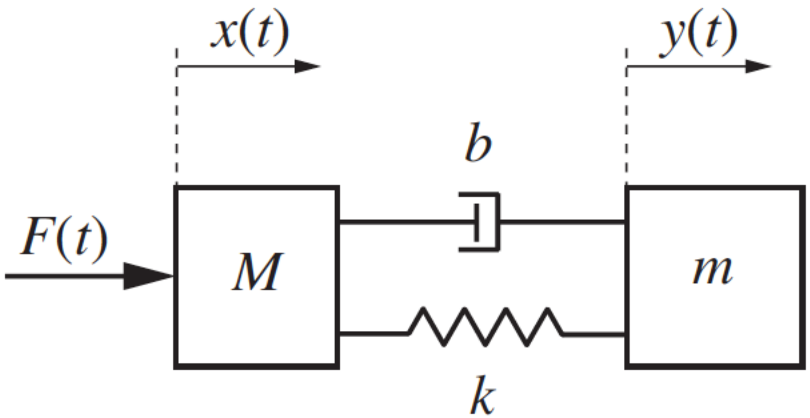
Equations of Motion

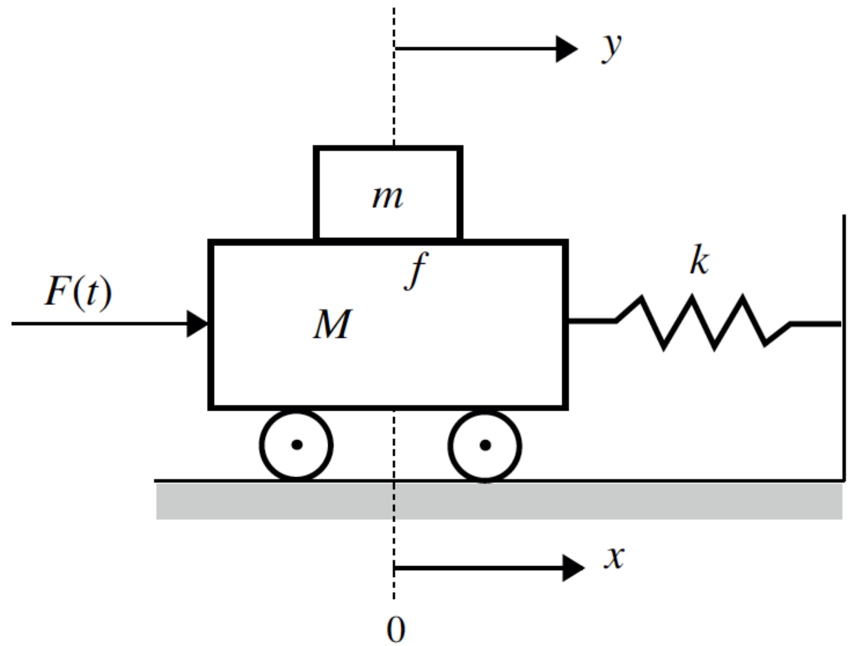
$$m\frac{d^2x_1}{dt^2} = -kx_1 - k(x_1 - x_2) - f\frac{dx_1}{dt}$$

$$m\frac{d^2x_2}{dt^2} = -kx_2 - k(x_2 - x_1) - f\frac{dx_2}{dt}$$

- Both x_1 and x_2 are necessary to the describe the motion of all the elements in the system
- There is no input: self-driven

Example 4: Application of External Forces





$$M\frac{d^2}{dt^2}x(t) = F(t) - b\frac{d}{dt}(x-y) - k(x-y)$$

$$m\frac{d^2}{dt^2}y(t) = b\frac{d}{dt}(x-y) + k(x-y)$$

$$M\ddot{x} = -kx - f(\dot{x}-\dot{y}) + F$$

$$m\ddot{y} = f(\dot{x}-\dot{y})$$

Elements of Electrical Systems

- Input from voltage and current sources

- Basic characteristics
  - Potential
  - Charge
  - Current

v: [V] Volt

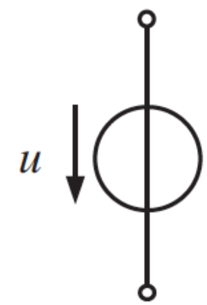
q: [C] Coulomb

i: [A] = [C/s] Ampere

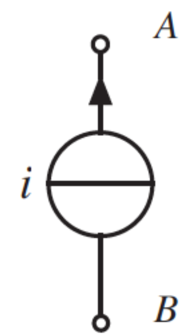
- Basic building blocks
  - Inductance elements
  - Capacitance elements
  - Resistive elements

- Inductance and capacitance elements store energy
- Resistive elements dissipate energy

Voltage Source

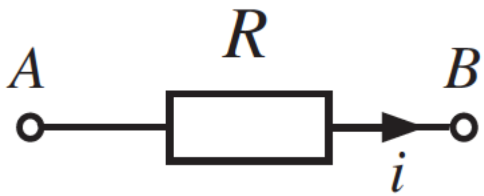


Current Source



Circuit Elements

Resistor

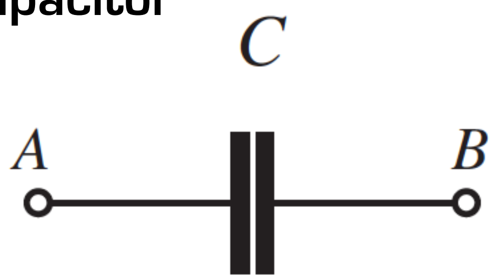


Electric potential difference (Voltage)

$$v_A - v_B = u_{AB} = Ri \qquad R:\ [\Omega] = \left[\frac{V}{A}\right]$$

Ohm

Capacitor



$$q = C(v_A - v_B) = Cu_{AB}$$

$$i = \frac{dq}{dt} = C\,\frac{du_{AB}}{dt} \qquad C:\ [F] = \left[\frac{C}{V}\right]$$

Farad

Inductor

$$v_A - v_B = u_{AB} = L\,\frac{di}{dt} \qquad L = [H] = \left[\frac{Vs}{A}\right]$$

Henry

Practical Considerations

- Real resistors are nonlinear and exhibit some capacitance and inductance

- **Self-inductance:** magnetic field generated by the coil induces voltage. The magnitude of the induced voltage is proportional to the rate of change of flux. Without an iron core, flux is proportional to di/dt.
- **Mutual inductance:** Influence between inductors due to generated fields

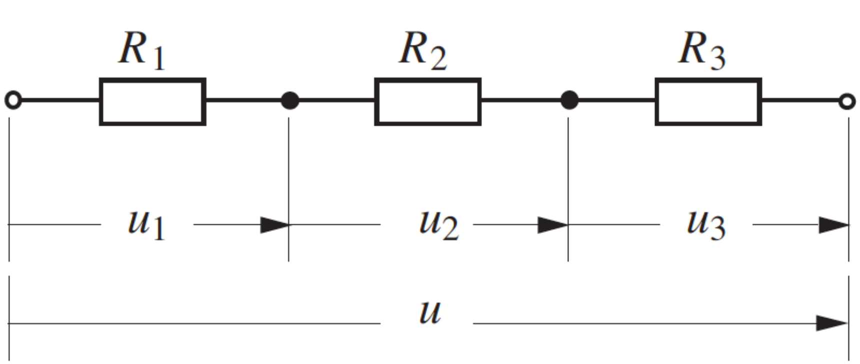
- Most inductors are coils of wire, that have considerable resistance
  - **Quality factor**: Ratio of stored to dissipated energy

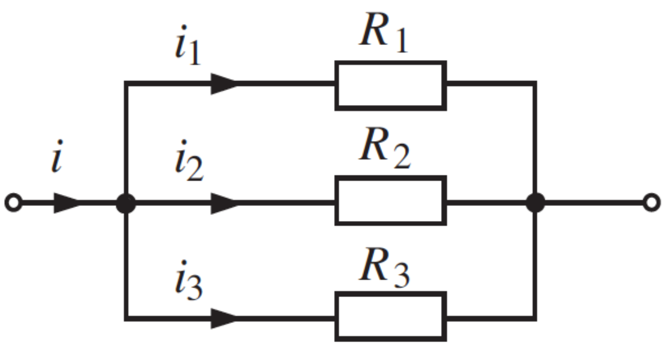
- Two metallic plates separated by a very thin dielectric medium form a capacitor

- Real capacitors exhibit various losses.
  - **Power factor**: ratio of energy lost per cycle to energy stored per cycle

Ohm’s Law

- Current in a circuit is proportional to the total electromotive force and inversely proportional to the total resistance
- Equivalent resistance (series and parallel arrangements)





$$R = R_1 + R_2 + R_3$$

$$u = u_1 + u_2 + u_3$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

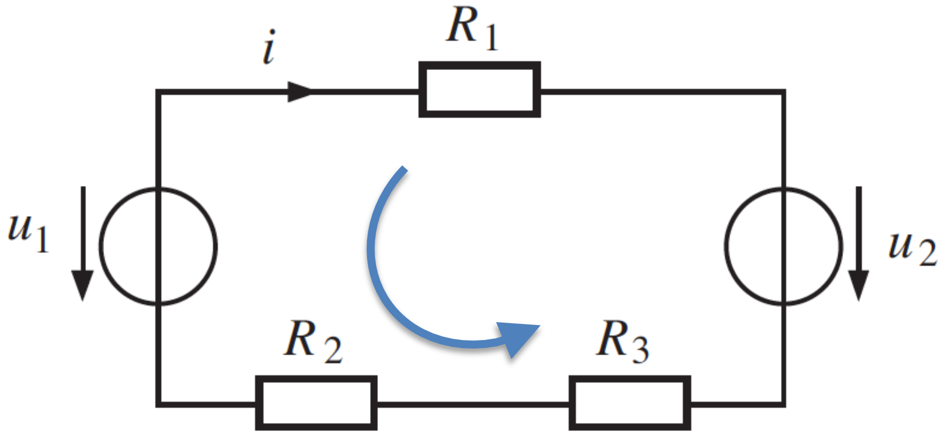
$$i = i_1 + i_2 + i_3$$

Kirchhoff's Law

- The sum of all currents entering a node is equal to the sum of all the currents leaving the same node

- The sum of all the voltage drops is equal to the sum of all the voltage rises around a loop

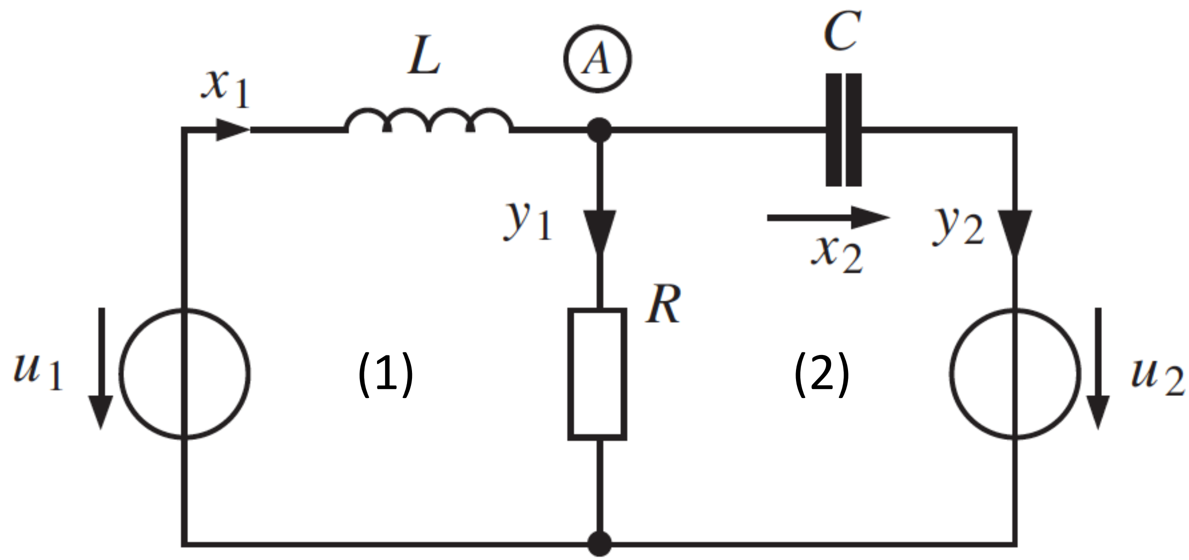
- First step is to define the directions of currents in each wire
- Second step is to determine the directions we follow in each loop



$$iR_1 + u_2 + iR_3 + iR_2 - u_1 = 0$$

$$i = \frac{u_1 - u_2}{R_1 + R_2 + R_3}$$

Example: RLC Circuit



- Inputs: voltages u_1 and u_2
- Outputs: currents y_1 and y_2

- x_1 : current flowing through inductor
- x_2 : voltage on the capacitor

Example: RLC Circuit

- The sum of voltages in the first loop is zero

$$L \frac{\mathrm{d}x_1}{\mathrm{d}t} + Ry_1 = u_1$$

- The sum of voltages in the second loop is zero

$$x_2 + u_2 = Ry_1 \quad \Rightarrow \quad y_1 = \frac{1}{R}(x_2 + u_2)$$

$$\Downarrow$$

- The sum of currents at node A is zero

$$x_1 - y_1 - y_2 = 0 \quad \Rightarrow \quad y_2 = x_1 - \frac{1}{R}(x_2 + u_2)$$

- At the capacitor

$$y_2 = C \frac{\mathrm{d}x_2}{\mathrm{d}t}$$

Example: RLC Circuit

- Combine all equations together:

$$L\frac{\mathrm{d}x_1}{\mathrm{d}t} + x_2 + u_2 - u_1 = 0$$

$$C\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 - \frac{1}{R}(x_2 + u_2)$$

$\Rightarrow$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{1}{L}(u_1 - x_2 - u_2)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{1}{C}x_1 - \frac{1}{RC}(x_2 + u_2)$$

Example: RLC Circuit

- Matrix Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ 0 & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{R} \\ 1 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{R} \\ 0 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Analogous Systems

- Physically different systems described by mathematical equations of identical form.
- The solution of one system can inform the other

- Electrical systems are easier to experimentally construct so their models are easier to validate

Analogies Between Mechanical and Electrical Systems

- Force-Voltage Analogy

Mechanical System	Electrical System
Force F and Torque M	Voltage u
Mass m and Moment of Inertia J	Inductance L
Viscous-friction f	Resistance R
Spring Constant k	Reciprocal of Capacitance $1/C$
Displacement x or θ	Charge q
Velocity v or w	Current i

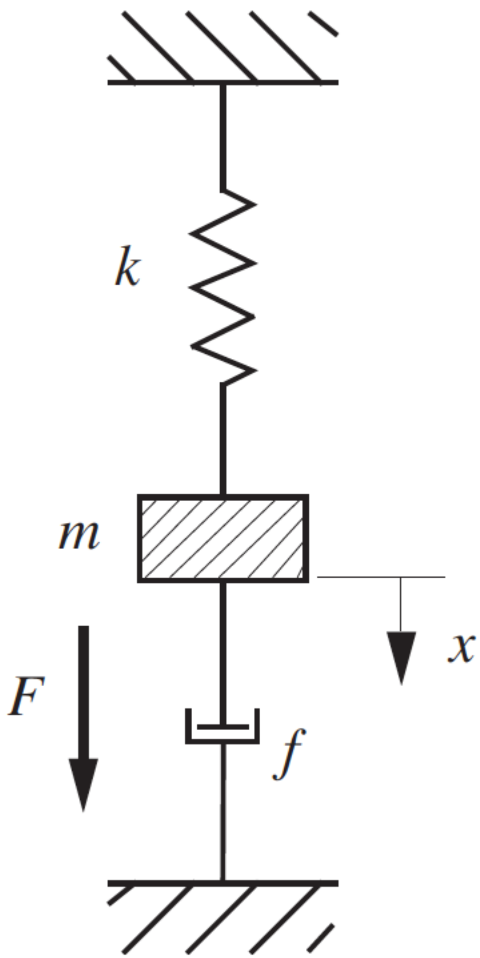
- Mechanical and electrical elements in series

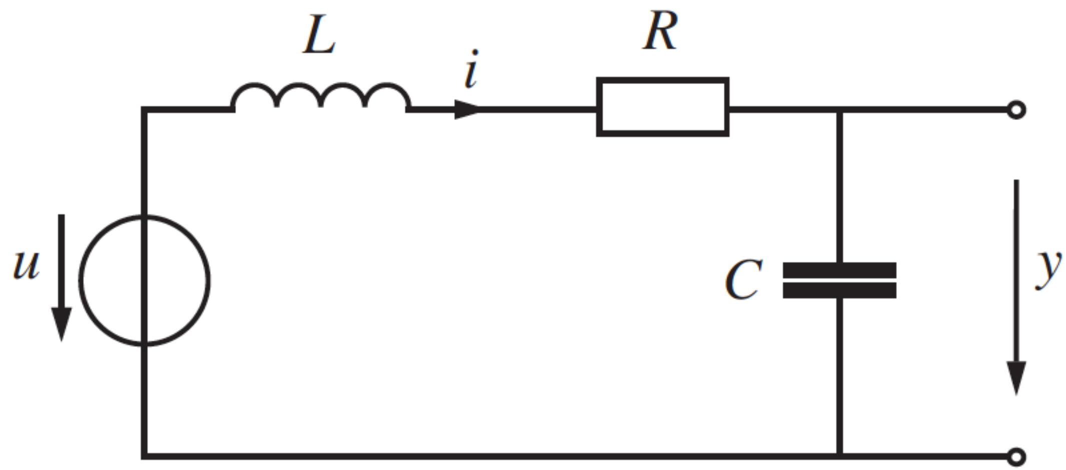
$$\frac{1}{f_t} = \frac{1}{f_1} + \frac{1}{f_2} \qquad R_t = R_1 + R_2$$

$$\frac{1}{k_t} = \frac{1}{k_1} + \frac{1}{k_2} \qquad \frac{1}{C_t} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$m_t = m_1 + m_2 \qquad L_t = L_1 + L_2$$

Example 1



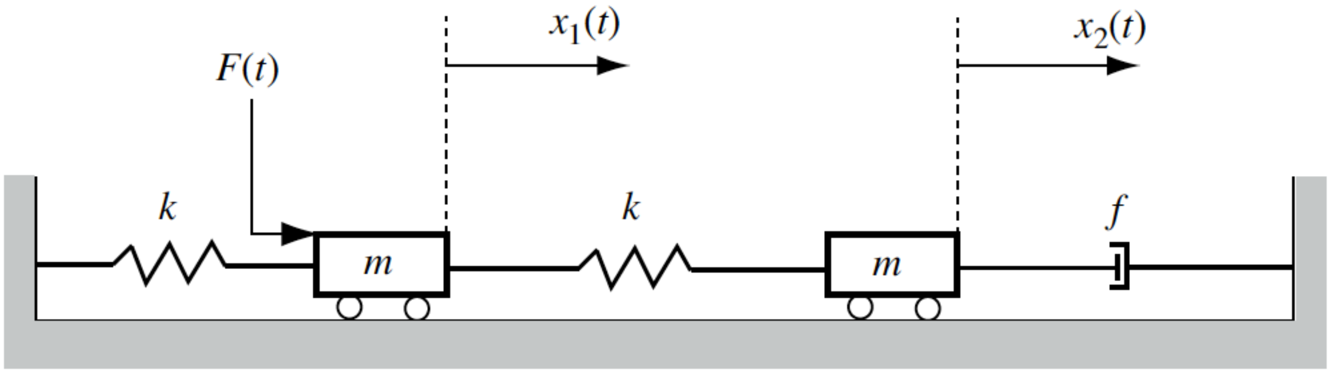


Equations

$$m\ddot{x}(t) + f\dot{x}(t) + kx(t) = F(t) \qquad x(0) = x_0 \;\; \dot{x}(0) = v_0$$

$$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = u(t) \qquad q(0) = q_0 \;\; \dot{q}(0) = i_0$$

Example 2



$$m\ddot{x}_1 = F - kx_1 - k(x_1 - x_2)$$

$$m\ddot{x}_2 = k(x_1 - x_2) - f\dot{x}_2$$